MATHEMATICS

THE COEFFICIENT OF THE S-FUNCTION $\{nm-k-r, k, r\}$, $k \leqslant m$, IN THE ANALYSIS OF $\{m\} \otimes \{\nu\}$, WHERE (ν) IS ANY PARTITION OF n, AND $n=5$ OR 6. I

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1. *Introduction*

When (μ) and (ν) are partitions of m and n respectively, the (ν) invariant matrix of the (μ) invariant matrix of a given square matrix A, is an invariant matrix of A, which is in general reducible, that is, it is equivalent to the direct sum of (λ) invariant matrices of A, where (λ) is a partition of mn. Thus

$$[A^{(\mu)}]^{(\nu)} \doteq \sum k_{\mu\nu\lambda} A^{(\lambda)}, \tag{1}$$

where $k_{\mu\nu\lambda}$ is a scalar number. The characteristic roots of $A^{(\mu)}$ are monomials of degree m in the characteristic roots of A. The sum of these monomials, i.e. the spur of $A^{(\mu)}$ is the S-function $\{\mu\}$ in the characteristic roots of A. The S-function $\{\nu\}$ in the characteristic roots of $A^{(\mu)}$ is a homogeneous symmetric function of degree mn in the characteristic roots of A. According to the equivalence (1) this symmetric function is the sum of the S-functions $\{\lambda\}$, of weight mn, in the characteristic roots of A. LITTLEWOOD writes this association in the form

$$\{\mu\} \otimes \{\nu\} = \sum k_{\mu\nu\lambda}\{\lambda\}, \tag{2}$$

and he calls it the plethysm of the S-functions $\{\mu\}$ and $\{\nu\}$, in this order, since the operation is found to be non-commutative. This plethysm of S-functions is found to be of central importance in algebraic invariant theory. Thus [6] the equality (2) indicates that there are $k_{\mu\nu\lambda}\, g_{\tau\nu}$ concomitants of type $\{\lambda\}$ and class $\{\nu\}$ which are of degrees $\tau_1, \tau_2, \ldots, \tau_i$ respectively in i ground forms each of type $\{\mu\}$, where $(\tau) = (\tau_1, \tau_2, \ldots, \tau_i)$ is a partition of n and $g_{\tau\nu}$ is the coefficient of $\{\nu\}$ in the product $\{\tau_1\}\{\tau_2\} \ldots \{\tau_i\}$. Further, this plethysm of S-functions has an obvious interpretation in the theory of the representations of the full linear group, and also an interpretation, due to ROBINSON [13], in the theory of matrix representations of the symmetric group.

When the matrix A is of order $r \times r$, $r < n$, or when the ground forms of type $\{\mu\}$ involve r variables only, then every S-function consisting of more than r parts is identically zero and we have what is called the r-ary analysis of $\{\mu\} \otimes \{\nu\}$. The complete analysis of $\{m\} \otimes \{\nu\}$ contains S-functions of not more than n parts.

Several methods have been devised for the evaluation of the plethysm of S-functions. In particular we have LITTLEWOOD's methods [5, 8], MURNAGHAN's treatment [10, 11, 12], ROBINSON's method [14] and TODD's method [16]. All methods are effective when the product mn is small, the range of effectiveness varying from one method to another. Further, we have FOULKES treatment with differential operators [3] which is theoretically capable of finding the coefficient of any S-function in any $\{\mu\} \otimes \{\nu\}$.

It is interesting to have some explicit formulae, even for the coefficients of S-functions $\{\lambda\}$ of some specified types. THRALL [15] gives explicit fomulae for $\{m\} \otimes \{2\}$ and $\{m\} \otimes \{3\}$, and similar formulae can be written for $\{m\} \otimes \{1^2\}$, $\{m\} \otimes \{21\}$ and $\{m\} \otimes \{1^3\}$. DUNCAN [1] and FOULKES [4] independently have investigated the analysis of $\{m\} \otimes \{\nu\}$, where (ν) is any partition of 4. But FOULKES [4, pp. 575–576] has given some explicit formulae which enable us to write down all S-functions in the ternary analysis of $\{m\} \otimes \{\nu\}$, for which the second part is $\leqslant m$; these formulae are remarkable in that they are independent of m. We give here formulae which enable us to write down all S-functions in the ternary analysis of $\{m\} \otimes \{\nu\}$, for which the second part is $\leqslant m$, where (ν) is any partition of 5 or 6. The treatment is by the use of FOULKES differential operators. Writing an S-function of weight mn and of three parts in the form $\{nm-k-r, k, r\}$, we include tables of coefficients of such S-functions, for all $0 \leqslant r \leqslant k \leqslant 12$, in the analysis of $\{m\} \otimes \{n\}$ and $\{m\} \otimes \{1^n\}$ where $n=5$ or 6 and provided $m \geqslant k$.

2. *Differential operators associated with S-functions*

With the S-function $\{\mu\}=\{\mu_1, \mu_2, \ldots, \mu_p\}$, $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_p > 0$, written in the JACOBI-TRUDI form

(3) $$\{\mu\}=|\{\mu_i-i+j\}|$$

where i is a row-suffix and j is a column suffix, FOULKES [3] associates a differential operator D_μ, which written in determinantal form is

(4) $$D_\mu=|D_{\mu_i-i+j}|.$$

If (μ) is a partition of m and $(\nu)=(\nu_1, \nu_2, \ldots, \nu_q)$, $\nu_1 \geqslant \nu_2 \geqslant \ldots \geqslant \nu_q > 0$ is a partition of n, and if $m \leqslant n$ and $p \leqslant q$, then $D_\mu\{\nu\}$ is the general isobaric determinant

(5) $$D_\mu\{\nu\}=\{\nu/\mu\}$$

which is obtained from the JACOBI-TRUDI determinant of $\{\nu\}$ by subtracting μ_1 from the partitions (of single parts) in the first column, μ_2 from the partitions in the second column, and so on. In particular if $m=n$, $D_\mu\{\nu\}=1$ if $(\mu)=(\nu)$ and $D_\mu\{\nu\}=0$ if $(\mu) \neq (\nu)$. Also if $m>n$ or $p>q$ then $D_\mu\{\nu\}=0$.

Frequently we have to consider $D_\mu\{\nu\}$ as the result of operating with the S-function $\{\nu\}$, in the reverse sense, on the differential operator D_μ. In this case if $m \geqslant n$ and $p \geqslant q$, then

$$D_\mu\{\nu\} = D_{\mu/\nu}\,, \tag{6}$$

but if $m<n$ or $p<q$, then $D_\mu\{\nu\}=0$.

An important remark [3] is that differential operators follow the same multiplication rules as the corresponding S-functions, i.e.,

$$D_\xi D_\eta = \sum g_{\xi\eta\zeta} D_\zeta \text{ whenever } \{\xi\}\{\eta\} = \sum g_{\xi\eta\zeta}\{\zeta\}. \tag{7}$$

The S-function $\{\mu\}$ written as the simple characteristic, corresponding to the partition (μ), of the symmetric group G_m is, [7, p. 86],

$$\{\mu\} = \frac{1}{m!}\sum_{(\alpha)} N_\alpha\, \chi^\mu_\alpha\, S_1^{\alpha_1} S_2^{\alpha_2} \ldots \tag{8}$$

where S_i is the usual sum of i^{th} powers, N_α is the number of elements of G_m in the class $(\alpha)=(1^{\alpha_1}\, 2^{\alpha_2} \ldots)$ and χ^μ_α is the character of that class, in the irreducible matrix representation of G_m, corresponding to the partition (μ). The associated differential operator [3] is

$$D_\mu = \frac{1}{m!}\sum_{(\alpha)} N_\alpha\, \chi^\mu_\alpha \cdot 1^{\alpha_1}\, 2^{\alpha_2} \ldots \frac{\partial^{\alpha_1+\alpha_2+\ldots}}{(\partial S_1)^{\alpha_1}(\partial S_2)^{\alpha_2}\ldots}. \tag{9}$$

When each S_i on the right hand side of (8) is replaced by S_{ri} we get a symmetric function of weight mr which is written, by FOULKES, $\{\mu\}^{(r)}$. FOULKES associates with $\{\mu\}^{(r)}$, the differential operator

$$D_\mu^{(r)} = \frac{1}{m!}\sum_{(\alpha)} N_\alpha\, \chi^\mu_\alpha \cdot r^{\alpha_1}(2r)^{\alpha_2} \ldots \frac{\partial^{\alpha_1+\alpha_2+\ldots}}{(\partial S_r)^{\alpha_1}(\partial S_{2r})^{\alpha_2}\ldots}. \tag{10}$$

Both $\{\mu\}^{(r)}$ and $D_\mu^{(r)}$ follow the same multiplication rules as the corresponding S-functions. Thus corresponding to (7) we have

$$\{\xi\}^{(r)}\{\eta\}^{(r)} = \sum g_{\xi\eta\zeta}\{\zeta\}^{(r)}, \tag{11}$$

$$D_\xi^{(r)} D_\eta^{(r)} = \sum g_{\xi\eta\zeta} D_\zeta^{(r)}. \tag{12}$$

The effect of $D_\mu^{(r)}$ on $\{\nu\}^{(r)}$ is similar to the effect of D_μ on $\{\nu\}$, in particular when (μ) and (ν) are partitions of the same number, then

$$D_\mu^{(r)}\{\nu\}^{(r)} = 1 \text{ if } (\mu)=(\nu)\,; = 0 \text{ if } (\mu)\neq(\nu). \tag{13}$$

FOULKES [2, 3 and 4] gives the following method for evaluating $D_\tau[\{\mu\}^{(r)}]^s$, where (τ) is a partition of mrs. First, in the JACOBI-TRUDI form of $\{\tau\}$ each $\{\tau_i - i + j\}$ is replaced by $\{\tau_i - i + j/r\}^{(r)}$ when $(\tau_i - i + j) \equiv 0 \pmod r$, and by zero when $(\tau_i - i + j) \neq 0 \pmod r$. Applying the LITTLEWOOD-RICHARDSON rule for the ordinary multiplication of S-functions [7, p. 94, V], and making use of the correspondence (12), the resulting determinant is written as the sum of operators $\Sigma_\sigma D_\sigma^{(r)}$ where (σ) is a partition of ms. The required result is then obtained by evaluating

$\Sigma_\sigma D_\sigma[\Sigma_{\sigma'}\{\sigma'\}]$, according to the rule (13), where $\Sigma\{\sigma'\}=\{\mu\}^s$. In particular, when $s=1$, $D_\tau\{m\}^{(r)}$ is the value of the determinant obtained by replacing, in the JACOBI-TRUDI form of $\{\tau\}$, each $\{\tau_i-i+j\}$ by 1 when $(\tau_i-i+j)\equiv 0 \pmod r$ and by 0 when $(\tau_i-i+j)\not\equiv 0 \pmod r$.

3. *The coefficient of* $\{5m-k-r, k, r\}$, $0\leqslant r\leqslant k\leqslant m$, *in the analysis of* $\{m\}\otimes\{\nu\}$, *where* (ν) *is any partition of* 5

When (ν) is a partition of 5, then [3, § 5 or 16, § 4]

$$(14)\quad \begin{cases} \{m\}\otimes\{\nu\}=\frac{1}{5!}[\chi^\nu_{1^5}\{m\}^5+10\,\chi^\nu_{1^32}\{m\}^3\{m\}^{(2)} \\ +20\,\chi^\nu_{1^23}\{m\}^2\{m\}^{(3)}+15\,\chi^\nu_{12^2}\{m\}\,(\{m\}^{(2)})^2+30\,\chi^\nu_{14}\{m\}\,\{m\}^{(4)} \\ +20\,\chi^\nu_{23}\{m\}^{(2)}\{m\}^{(3)}+24\,\chi^\nu_{5}\{m\}^{(5)}]. \end{cases}$$

The coefficient of the S-function $\{\lambda\}$, of weight $5m$, in the analysis of $\{m\}\otimes\{\nu\}$ is given by $D_\lambda\{m\}\otimes\{\nu\}$. The number $D_\lambda\{m\}\otimes\{\nu\}$ may be obtained by finding the result of operating with D_λ on each separate term on the right hand side of (14) and then gathering the results, making use of the character table of the symmetric group of degree 5. In the following, we obtain the result of operating with D_λ on those terms, where $(\lambda)=(5m-k-r, k, r)$, $0\leqslant r\leqslant k\leqslant m$.

(I) To obtain $D_\lambda\{m\}^5$ we simply apply one of FOULKES important results [4, th. 2]. As we shall frequently make use of this result we state it in the following lemma:

Lemma: The coefficient of the S-function $\{\nu\}=\{\nu_1, \nu_2, \ldots, \nu_n\}$, $\nu_1\geqslant\nu_2\geqslant\ldots\geqslant\nu_n\geqslant 0$, $\nu_2\leqslant m$, in $\{m\}^n$ is

$$\prod_{r<s}[(\nu_r-r)-(\nu_s-s)]\,\big/\,(n-2)!\ (n-3)!\ldots 1!$$

where r, s take all values from 2 to n.

Applying this result we have

$$\underline{(i)\ D_\lambda\{m\}^5}=\frac{1}{12}(k+2)\,(k+3)\,(k-r+1)\,(r+1)\,(r+2).$$

(II) To obtain $D_\lambda\{m\}^3\{m\}^{(2)}$ we write

$$D_\lambda\{m\}^3\{m\}^{(2)}=\begin{vmatrix} D_{5m-k-r} & D_{5m-k-r+1} & D_{5m-k-r+2} \\ D_{k-1} & D_k & D_{k+1} \\ D_{r-2} & D_{r-1} & D_r \end{vmatrix}\{m\}^3\cdot\{m\}^{(2)},$$

that is we first operate with $\{m\}^3$, in the reverse sense, on D_λ. Since $\{m\}^3$ contains S-functions, of weight $3m$, of not more than three parts, we may write

$$\{m\}^3=\sum_{u,v}a_{u,v}\{3m-u-v, u, v\}.$$

Operating with $\{3m-u-v, u, v\}$, where $u>k$, on D_λ the resulting minor in the second and third rows and the first and second columns consists of zero elements, since the subscripts become negative. Thus $D_\lambda\{3m-u-v, u, v\}=0$, for all $u>k$. Hence we have to evaluate

$$D_\lambda [\sum_{u,v} a_{u,v} \{3m-u-v, u, v\}] \cdot \{m\}^{(2)},$$

where $0 \leqslant v \leqslant u \leqslant k$, and by the above lemma $a_{u,v}=(u-v+1)$. Since $k \leqslant m$, then $3m-u-v \geqslant m$, so that this summation becomes

$$\sum_{u,v} (u-v+1) D_{2m-k-r+u+v} \begin{vmatrix} D_{k-u} & D_{k+1-v} \\ D_{r-1-u} & D_{r-v} \end{vmatrix} \{m\}^{(2)}. \tag{15}$$

Now, four cases arise which are as follows.

(a) *k and r are both even, say* $k=2l$ and $r=2s$

When u and v are of different parities the corresponding term is evidently zero. When u and v are both even, say $u=2p$ and $v=2q$, the corresponding term is

$$D_{m-l-s+p+q}\, D_{l-p}\, D_{s-q}\, \{m\}$$

$=1$ whenever $p \leqslant l$ and $q \leqslant \min(p, s)$. When u and v are both odd say $u=2p+1$ and $v=2q+1$, the corresponding term is

$$-D_{m-l-s+p+q+1}\, D_{s-p-1}\, D_{l-q}\, \{m\}$$

$=-1$ whenever $p \leqslant s-1$ and $q \leqslant p$. Hence the summation in (15) is

$$\begin{aligned}
&= [\sum_{p=0}^{s} \sum_{q=0}^{p} + \sum_{p=s+1}^{l} \sum_{q=0}^{s}] (2p-2q+1) \\
&\quad - \sum_{p=0}^{s-1} \sum_{q=0}^{p} (\overline{2p+1} - \overline{2q+1} + 1) \\
&= \sum_{p=s}^{l} \sum_{q=0}^{s} (2p-2q+1) \\
&= (l+1)(l-s+1)(s+1) \\
&= \tfrac{1}{8}(k+2)(k-r+2)(r+2).
\end{aligned}$$

(b) *k is even, say $k=2l$ and r is odd, say* $r=2s+1$

When u and v are both even, u and v are both odd, or u is odd and v is even, the corresponding term is always zero. A term which may not be zero corresponds to an even u say $u=2p$ and an odd v say $v=2q+1$. In such a case the term is

$$D_{m-l-s+p+q} \begin{vmatrix} D_{l-p} & D_{l-q} \\ D_{s-p} & D_{s-q} \end{vmatrix} \{m\} \quad .$$

so that the summation in (15) is

$$\begin{aligned}
&= \sum_{p=s+1}^{l} \sum_{q=0}^{s} (2p - \overline{2q+1} + 1) \\
&= (l+1)(l-s)(s+1) \\
&= \tfrac{1}{8}(k+2)(k-r+1)(r+1).
\end{aligned}$$

(c) *k is odd, say* $k=2l+1$ *and r is even, say* $r=2s$

A term which may not be zero corresponds to an odd u, say $u=2p+1$ and an even v, say $v=2q$. In such a case, the term is

$$D_{m-l-s+p+q}\begin{vmatrix} D_{l-p} & D_{l-q+1} \\ D_{s-p-1} & D_{s-q} \end{vmatrix}\{m\}$$

so that the summation in (15) is

$$\begin{aligned} &= \sum_{p=s}^{l}\sum_{q=0}^{s}(\overline{2p+1}-2q+1) \\ &= (l+2)\,(l-s+1)\,(s+1) \\ &= \tfrac{1}{8}\,(k+3)\,(k-r+1)\,(r+2). \end{aligned}$$

(d) *k and r are both odd, say* $k=2l+1$ *and* $r=2s+1$

This case is similar to case (a), but positive contributions occur when both u and v are odd, and negative contributions occur when both u and v are even. The summation in (15), in this case, is

$$\begin{aligned} &= [\sum_{p=0}^{s}\sum_{q=0}^{p}+\sum_{p=s+1}^{l}\sum_{q=0}^{s}]\,(\overline{2p+1}-\overline{2q+1}+1) \\ &- \sum_{p=0}^{s}\sum_{q=0}^{p}(2p-2q+1) \\ &= \sum_{p=s+1}^{l}\sum_{q=0}^{s}(2p-2q+1) \\ &= (l+2)\,(l-s)\,(s+1) \\ &= \tfrac{1}{8}\,(k+3)\,(k-r)\,(r+1). \end{aligned}$$

The following table gathers the above four results:

(ii) $8D_\lambda\{m\}^3\,\{m\}^{(2)}$

r/k	even	odd
even	$(k+2)\,(k-r+2)\,(r+2)$	$(k+3)\,(k-r+1)\,(r+2)$
odd	$(k+2)\,(k-r+1)\,(r+1)$	$(k+3)\,(k-r)\,(r+1)$.

(III) To obtain $D_\lambda\{m\}^2\,\{m\}^{(3)}$ we first operate with $\{m\}^2$, in the reverse sense, on D_λ. We make use of the equality ·

(16) $$\{m\}^2=\{2m\}+\{2m-1,\,1\}+\{2m-2,\,2\}+\ldots+\{m^2\}.$$

We get

(17) $$\sum_u D_{3m-k-r+u}\begin{vmatrix} D_{k-u} & D_{k+1} \\ D_{r-u-1} & D_r \end{vmatrix}\{m\}^{(3)}.$$

Nine cases arise according as $k,\,r=0,\,1$ or 2 (mod 3). All cases are quite simple, and applying the remarks in § 2, we get the following table of results:

(iii) $3D_\lambda\{m\}^2\{m\}^{(3)}$

r/k	0	1	2(mod 3)
0	$k+3$	$k+2$	$k-r+1$
1	0	0	$-(r+2)$
2(mod 3)	0	0	$-(r+1)$.

(IV) To obtain $D_\lambda\{m\}(\{m\}^{(2)})^2$, we first operate with $\{m\}$, in the reverse sense, on D_λ, to get

$$D_{4m-k-r}\begin{vmatrix} D_k & D_{k+1} \\ D_{r-1} & D_r \end{vmatrix}(\{m\}^{(2)})^2. \tag{18}$$

Then, there are four cases according to the parities of k and r. The two cases in which k and r are of different parities are trivial the answer in each case is evidently zero. The two cases, in which the parities of k and r are the same, are similar, and we illustrate with the case $k=2l$, $r=2s$. In this case (18) becomes

$$D_{2m-l-s}\,D_l\,D_s\,\{m\}^2.$$

According to (16) and the remarks in § 2, in the product of operators, on the left hand side, we need only to consider terms corresponding to partitions of not more than two parts. We now make use of (7) and the LITTLEWOOD–RICHARDSON rule for the multiplication of S-functions [7, p. 92, IV]. According to this rule, we have for two variables

$$\begin{aligned}\{2m-l-s\}\,\{l\}\,\{s\} &= \sum_{u=0}^{l}\{2m-l-s+u, l-u\}\,\{s\} \\ &= \sum_{u=0}^{l}\sum_{v=0}^{s}\{2m-l-s+u+v, l+s-u-v\}\end{aligned}$$

$$\begin{aligned}&=\{2m-l-s, l+s\}+2\,\{2m-l-s+1, l+s-1\}+3\,\{2m-l-s+2, l+s-2\} \\ &\quad+\ldots+s\,\{2m-l-1, l+1\}+(s+1)\,\{2m-l, l\}+(s+1)\,\{2m-l+1, l-1\} \\ &\quad+\ldots+(s+1)\,\{2m-s, s\}+s\,\{2m-s+1, s-1\}+\ldots+\{2m\}.\end{aligned}$$

Hence, evaluating $2\sum_{p=0}^{s}(p+1)+(l-s-1)\,(s+1)$, we get

$$D_{2m-l-s}\,D_l\,D_s\,\{m\}^2=(l+1)\,(s+1), \tag{19}$$

so that the result in the case under consideration is $\frac{1}{4}(k+2)\,(r+2)$.
The table gathering all cases is:

(iv) $4D_\lambda\,\{m\}\,(\{m\}^{(2)})^2$

r/k	even	odd
even	$(k+2)\,(r+2)$	0
odd	0	$-(k+3)\,(r+1)$

(V) To obtain $D_\lambda\{m\}\{m\}^{(4)}$, we first operate with $\{m\}$, in the reverse sense, on D_λ, then operate with the resulting operator on $\{m\}^{(4)}$; we simply get the following results:

(v) $\underline{D_\lambda\{m\}\{m\}^{(4)}} = 1$ when $k, r = 0 \pmod 4$,
$= -1$ when $k = 3 \pmod 4$ and $r = 1 \pmod 4$,
$= 0$ in all other cases.

(VI) To obtain $D_\lambda\{m\}^{(2)}\{m\}^{(3)}$, we first operate with

$$\{m\}^{(2)} = \{2m\} - \{2m-1, 1\} + \{2m-2, 2\} + \ldots + (-1)^m\{m^2\} \tag{20}$$

[10, p. 11], in the reverse sense, on D_λ to get the summation in (17) but with the term multiplied by $(-1)^u$. Apparently there are 9 cases as in (III), but the alternating sign of the term subdivides each case into four cases. The following table gathers all results:

(vi) $\underline{D_\lambda\{m\}^{(2)}\{m\}^{(3)}}$

r/k	0	1	2	3	4	5 (mod 6)
0	1	-1	1	0	0	0
1	0	0	-1	0	0	-1
2	0	0	1	0	0	1
3	1	-1	0	0	0	-1
4	0	0	0	0	0	0
5 (mod 6)	0	0	0	0	0	0.

(VII) The remarks in § 2, simply give the following table of results for $D_\lambda\{m\}^{(5)}$:

(vii) $\underline{D_\lambda\{m\}^{(5)}}$

r/k	0	1	2	3	4 (mod 5)
0	1	-1	0	0	0
1	0	1	0	0	-1
2	-1	0	0	0	1
3	0	0	0	0	0
4 (mod 5)	0	0	0	0	0.

Formula (14), the above results, and the character table of the symmetric group of degree 5 [7, p. 265 or 9, p. 142] enable us to get the coefficient of the S-function $\{5m-k-r, k, r\}$, for all $0 \leqslant r \leqslant k \leqslant m$, in the analysis of $\{m\} \otimes \{\nu\}$, where (ν) is any partition of 5. *Such coefficients are independent of the value of m.* We have obtained these coefficients for all partitions (ν) of 5, and for values of k and r such that $0 \leqslant r \leqslant k \leqslant 12$. We include here the tables for $(\nu) = (5)$ where all characters are unity, and $(\nu) = (1^5)$ where the characters are 1 for even classes and -1 for odd classes.

(A) Coefficient of $\{5m-k-r, k, r\}$ in $\{m\} \otimes \{5\}$, provided $m \geqslant k$

k/r	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	0	0											
2	1	0	1										
3	1	0	1	0									
4	2	1	3	2	3								
5	2	1	3	2	3	0							
6	3	3	6	6	8	5	7						
7	3	3	7	7	10	7	9	2					
8	5	5	11	12	17	15	19	12	12				
9	5	6	13	15	21	20	25	19	19	7			
10	7	9	18	22	31	32	40	35	38	27	22		
11	7	10	21	26	37	39	50	46	51	40	36	14	
12	10	14	28	36	51	56	71	70	79	71	70	49	38

The distinct values of k, in terms of which the results for a separate term of (14) are stated, differ from term to term so that compact direct formulae for the coefficient of $\{5m-k-r, k, r\}$ in $\{m\} \otimes \{\nu\}$ where (ν) is a partition of 5, are not feasible as is the case for the coefficient of $\{4m-k-r, k, r\}$ in $\{m\} \otimes \{\nu\}$, where (ν) is a partition of 4 [4, th. 25]. But if we assign to r a certain value, say $r=0, 1, 2, \ldots$ or k, that is if the S-function under consideration involves one parameter, k, then such direct compact formulae are feasible. The formulae for the coefficient of $\{5m-k, k\}$ are already obtained by Hopkins. We have obtained the formulae for the coefficient of $\{5m-2k, k^2\}$. We do not include these formulae here, we only exemplify, say, by the coefficient in $\{m\} \otimes \{32\}$, when $k=5$ or 11 (mod 12); the coefficient is $(k^4+8k^3+14k^2+8k+1)/288$.

(B) Coefficient of $\{5m-k-r, k, r\}$ in $\{m\} \otimes \{1^5\}$, provided $m \geqslant k$

k/r	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0												
1	0	0											
2	0	0	0										
3	0	0	0	0									
4	0	0	1	1	1								
5	0	0	0	1	1	0							
6	0	1	2	3	4	3	4						
7	0	1	2	4	5	5	6	2					
8	0	2	4	7	9	10	12	9	7				
9	0	2	5	9	12	14	17	15	14	7			
10	1	4	8	14	19	23	28	27	28	22	16		
11	0	5	9	17	23	29	36	37	39	35	29	14	
12	1	7	14	24	33	42	52	56	61	59	56	42	29

The coefficient of $\{5m-k-r, k, r\}$ in the ternary analysis of $\{m\} \otimes \{\nu\}$, is its coefficient in the complete analysis. This same coefficient is also the coefficient of some other S-functions in the analysis of $\{m\} \otimes \{\nu\}$ or $\{m\} \otimes \{\bar{\nu}\}$ according to FOULKES theorems on related coefficients [4, theorems 30, 36 and 33]. In the case under consideration FOULKES theorems take the form:

(a) The coefficient of the S-function $\{5m-4\alpha-k-r, k+\alpha, r+\alpha, \alpha^2\}$, $0 \leqslant \alpha \leqslant m-k$, in $\{m\} \otimes \{\nu\}$ is the same as the coefficient of $\{5m-k-r, k, r\}$ in $\{m\} \otimes \{\nu\}$ or $\{m\} \otimes \{\bar{\nu}\}$ according as α is even or odd.

(b) The coefficient of the S-function $\{5m-4\beta+k+r, \beta^2, \beta-r, \beta-k\}$, $k \leqslant \beta \leqslant m$, in $\{m\} \otimes \{\nu\}$ is the same as the coefficient of $\{5m-k-r, k, r\}$ in $\{m\} \otimes \{\nu\}$ or $\{m\} \otimes \{\bar{\nu}\}$ according as β is even or odd.

(c) The coefficient of the S-function $\{2m-\lambda_5, 2m-\lambda_4, \ldots, 2m-\lambda_1\}$, $\lambda_1 \leqslant 2m$, in $\{m\} \otimes \{\nu\}$ is the same as that of $\{\lambda_1, \lambda_2, \ldots, \lambda_5\}$.

For example the coefficient of the S-function $\{24, 4, 2\}$ in $\{6\} \otimes \{5\}$, given in table (A) furnishes us also with the coefficients of $\{16, 6, 4, 2^2\}$, $\{20, 4^2, 2\}$, $\{12, 6^2, 4, 2\}$ and $\{10, 8, 6^2\}$ in $\{6\} \otimes \{5\}$ and the coefficients of $\{20, 5, 3, 1^2\}$ and $\{16, 5^2, 3, 1\}$ in $\{6\} \otimes \{1^5\}$.

(To be continued)